Economics 352: Intermediate Microeconomics

Notes and Assignment

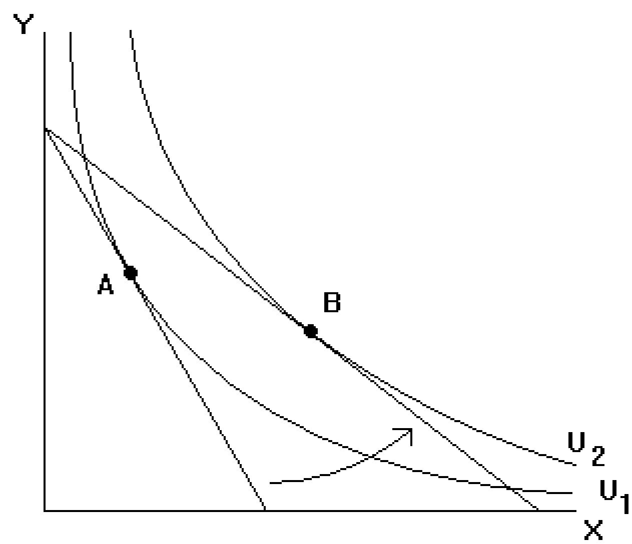
Chapter 5: Income and Substitution Effects

A Quick Introduction

To be clear about this, this chapter will involve looking at price changes and the response of a utility maximizing consumer to these price changes. The response of a consumer will be broken down into two parts: an *income effect* and a *substitution effect*.

Before things get unnecessarily complicated, I would like to lay these two parts out.

First, for a utility maximizing consumer a price change (a decrease in the price of good X, for example) actually looks like this:

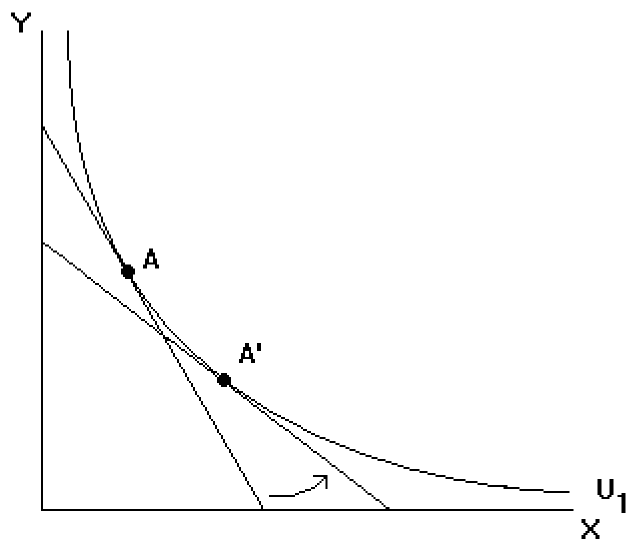


A graph showing the effect of a decrease in the price of good x on a consumers utility maximizing consumption decision.

When the price of X falls, the budget line rotates out and the consumer's utility maximizing bundle of goods changes from point A to point B, taking her from utility level U_1 up to utility level U_2.

Now, this move from A to be can be thought of as occurring in two parts. Alternatively, you might think about decomposing this move from A to B into two separate steps.

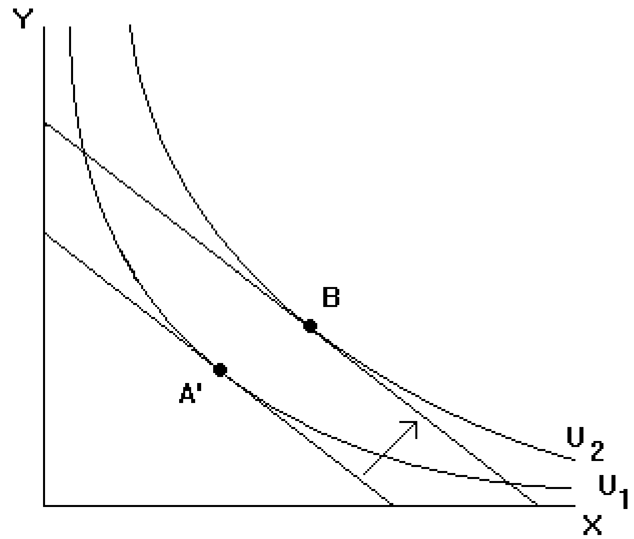
The first of these two parts is the substitution effect. The substitution effect reflects the idea that when the price of X changes the relative prices of X and Y change and the slope of the budget line changes. Put somewhat differently, the rate at which the consumer trades off X for Y changes. Expressed another way, the slope of the budget line is $-\frac{p_x}{p_y}$ and when the price of X, p_x, changes, this slope changes. The key to the substitution effect is that this change in the slope of the budget line is made holding the level of utility constant. In terms of the graph, the substitution effect is shown by rotating the original budget line around the initial indifference curve until it achieves its new slope:



A graph showing the substitution effect associated with a decrease in the price of good x.

The substitution effect moves the consumer from the bundle labeled A to the bundle labeled A'. The utility level remains at the original level U_1, but the change in the relative prices of X and Y means that the combination of X and Y changes. So, to be clear, the change in the bundle resulting from the substitution effect occurs because of the change in relative prices and if, as shown here, the price of X falls then more X will be consumed and less Y will be consumed.

Now, having discussed the substitution effect, let us turn to the income effect. Starting from the bundle A', the budget line shifts outward to the new budget line. This shift will be parallel and reflect the idea that when the price of X falls, the consumer's real income rises. That is, she is able to afford a larger set of bundles than she could afford previously. This income effect, the parallel shift, takes the consumer up to the new, higher utility level:



A graph showing the income effect of a decrease in the price of good x on a consumer's utility maximizing consumption decision.

So, the total effect of the decrease in the price of X is the move from point A to point B. This move can be decomposed into two parts. The move from A to A', the substitution effect, has no change in utility level and is only a result of the change in relative prices. This can be thought of as a rotation of the budget line around the original indifference curve, U_1. The move from A' to B, the income effect, takes the consumer to a new (higher) utility level and is a parallel shift, representing the increase in the set of affordable bundles that happens with the price of X falls.

Now, to some discussion of the chapter in the text.

Demand Functions

As shown in Chapter 4, it is possible to start with a utility function U=U(x,y) and an income constraint $I = p_x x + p_y y$ and from these calculate demand functions that give the quantities of x and y what will be demanded as functions of prices and income:

$$x^* = x(p_x, p_y, I)$$
$$y^* = y(p_x, p_y, I)$$

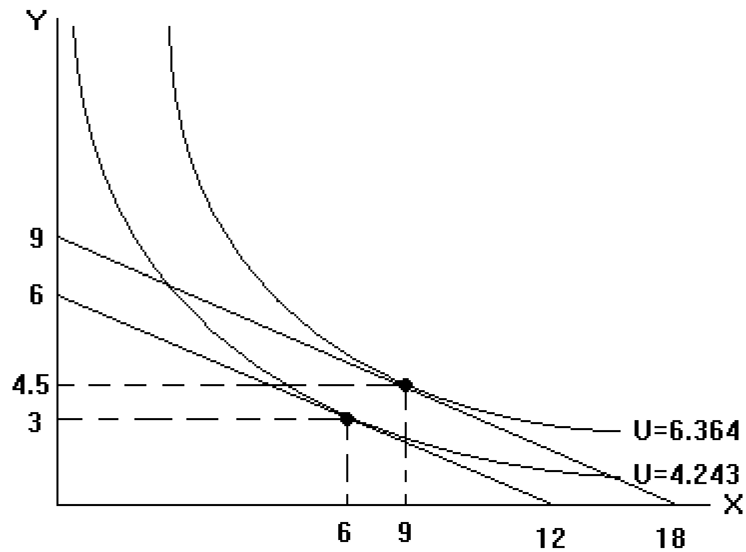
The concept of *homogeneity* says that if all prices and income are multiplied by the same number, then nothing changes. For example, if income doubled and all prices doubled as well, the same quantities of both goods would still be demanded and the resulting utility level wouldn't change. More specifically, we say that demand functions are homogeneous of degree zero in prices and income. Technically, this means:

$$x^* = x(p_x, p_y, I) = x(tp_x, tp_y, tI) = t^0 x(p_x, p_y, I) = x(p_x, p_y, I)$$
$$y^* = y(p_x, p_y, I) = y(tp_x, tp_y, tI) = t^0 y(p_x, p_y, I) = y(p_x, p_y, I)$$

That is, saying that the demand functions are homogeneous of degree zero means that multiplying all prices and income by t is equivalent to multiplying the value of the demand function by $t^0=1$. So, in the end, nothing changes.

Changes in Income

A change in income is represented in an indifference curve diagram as a parallel shift of the budget line. This is shown below for the situation where $U(x,y)=x^{0.5}y^{0.5}$, $p_x=1$, $p_y=2$ and income rises from 12 to 18:



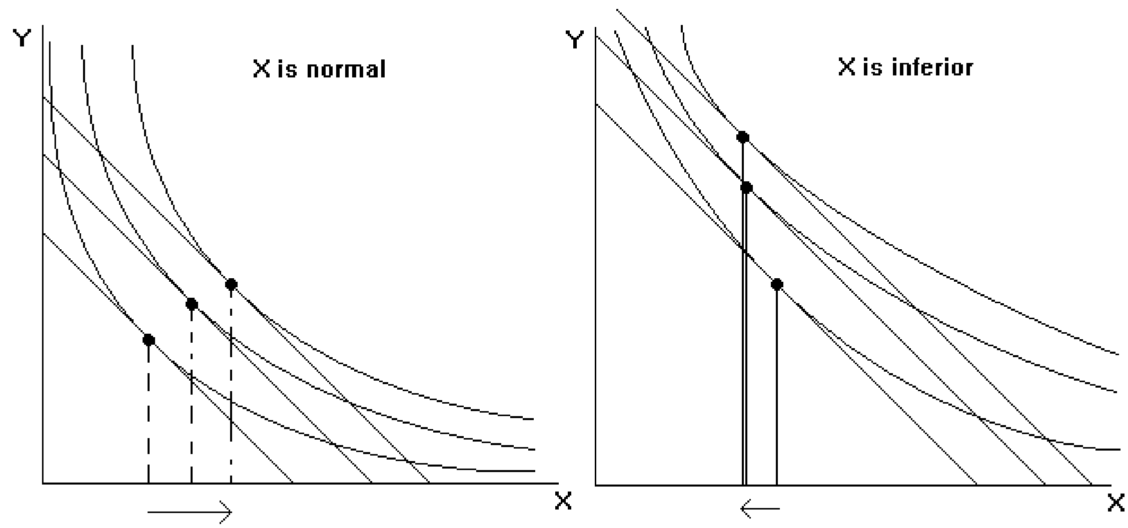
A graph showing the effect of a change in income from 12 to 18 for the above example.

You should confirm that the numbers shown here are correct.

When income increases and the budget line shifts out, consumption of any one good may either increase or decrease. If consumption of a particular good rises when income rises, this good is called a *normal good*. Normal goods are high quality things that you find very desirable and plan to consume more of as your income rises. If consumption of a particular good falls when income rises, this good is called an *inferior good*. Inferior goods are, perhaps, lower quality things that you expect to consume less of as your income rises. These might include low quality food and low quality housing.

It should be noted that the concepts of normal and inferior goods depend on the income level that a person starts with. A person with very low income might consider a good normal that you would consider inferior. A person with very high income might consider inferior a good that you consider normal.

In terms of diagram, this is how normal and inferior goods are represented:



Two graphs showing income expansion paths for two normal goods and for one normal good and one inferior good.

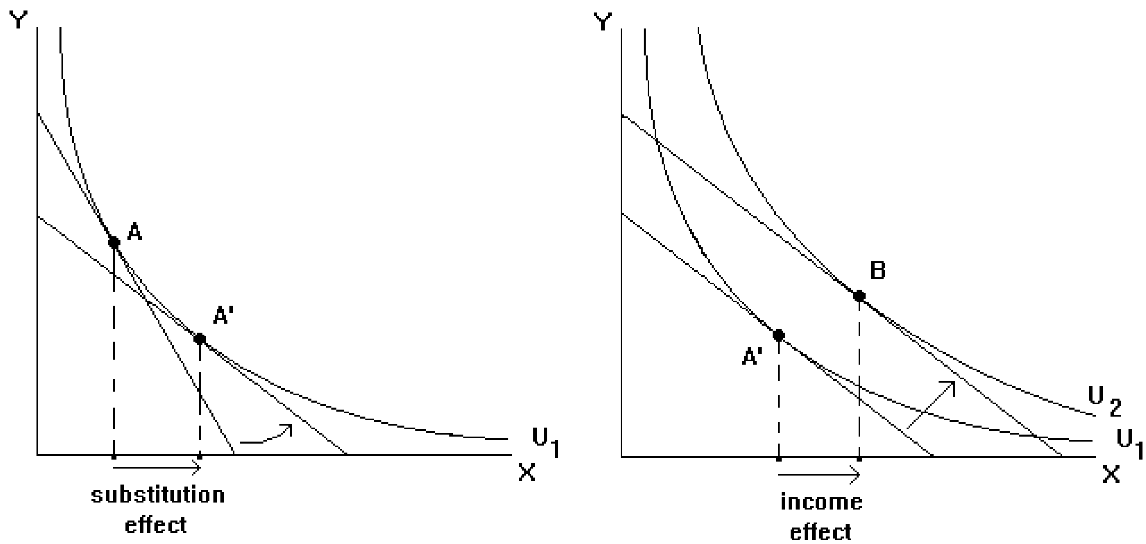
When X is normal, the quantity consumed increases as income increases. When X is inferior, the quantity consumed falls as income increases.

It should be noted that not all goods that a person consumes can be inferior. At least one good must be normal.

Changes in Price

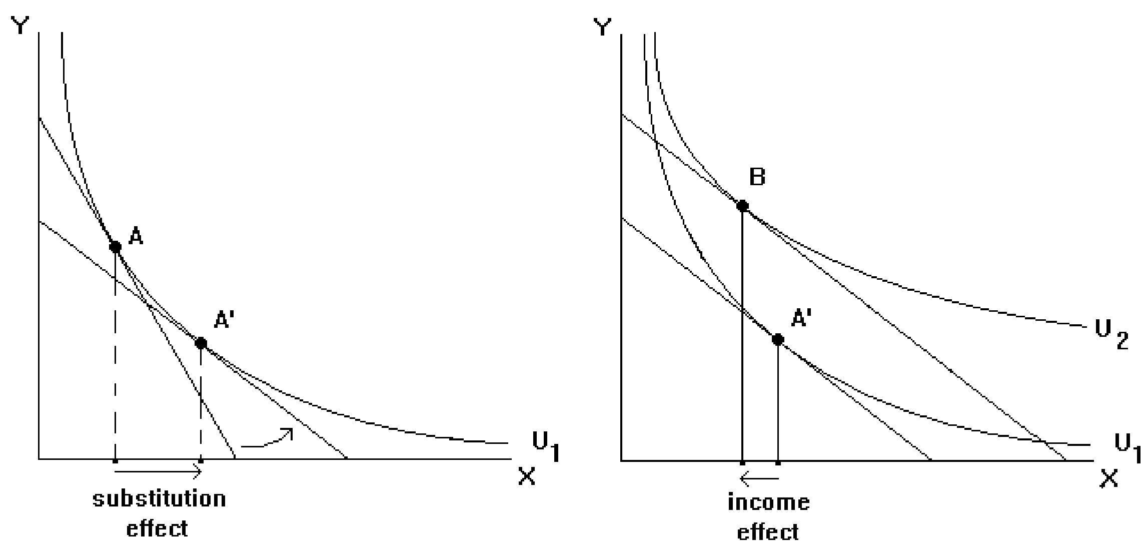
The analysis of changes in price presented in the book follows the discussion of income and substitution effects shown at the beginning of these lecture notes.

The additional point of interest is the decomposition of the change in the quantity of X consumed into substitution effects and income effects. From the above example, this can be shown as:



Two graphs showing the substitution effect of a decrease in the price of x and the income effect of a decrease in the price of x.

In this case, both the substitution and the income effects increase the quantity of X consumed. However, if X were an inferior good then the income effect would be negative. That is, the income effect would slightly reduce the quantity of X consumed:



Two graphs showing the substitution and income effects associated with a decrease in the price of x if x is an inferior good.

The net effect of the decrease in the price of X would still be an increase in the quantity of X consumed, but this net increase will not be as large as if X had been a normal good.

In the diagram shown here, the substitution effect is larger than the income effect, so the quantity of X consumed rises when the price of X falls.

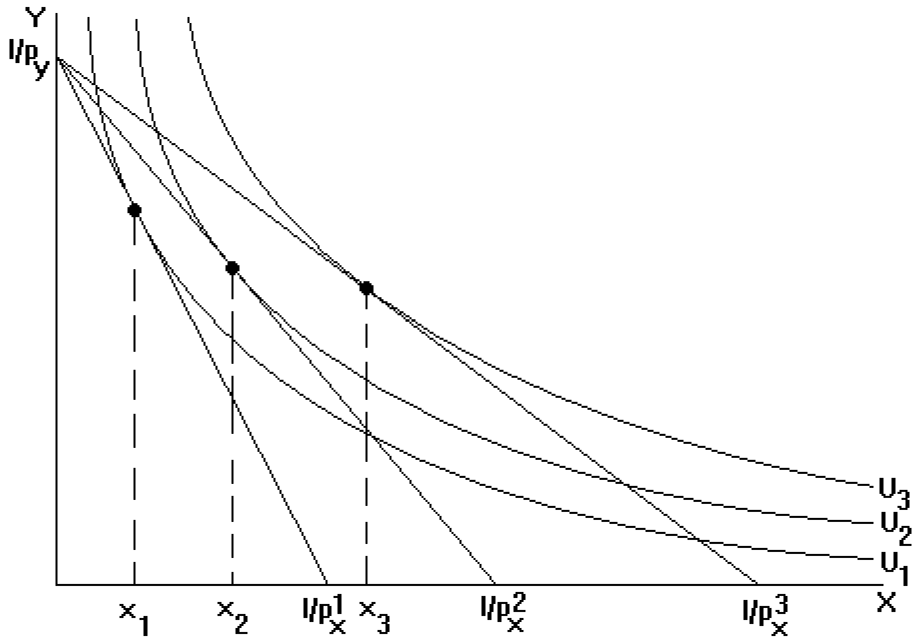
Put a slightly different way, if the substitution effect is larger than the income effect (if the substitution effect dominates the income effect) then the net result of a decrease in the price of X will be an increase in the quantity of X consumed, even if the income effect reduces the quantity of X consumed.

There is a bizarre, but theoretically possible case where the income effect outweighs the substitution effect. This is called a *Giffen good*, which the textbook describes under the heading Giffen's paradox. If the income effect is negative and outweighs or dominates the substitution effect, then it could be possible that a decrease in the price of X will lead to less, rather than more, X being consumed.

The standard example of a Giffen good is potatoes in nineteenth century Ireland. Potatoes were a staple of the people's diet and when their price rose people became much poorer in a real sense. These people substituted away from other, normal goods, and bought more of the relatively inferior potatoes, with the net effect that consumption of potatoes rose even as their price rose.

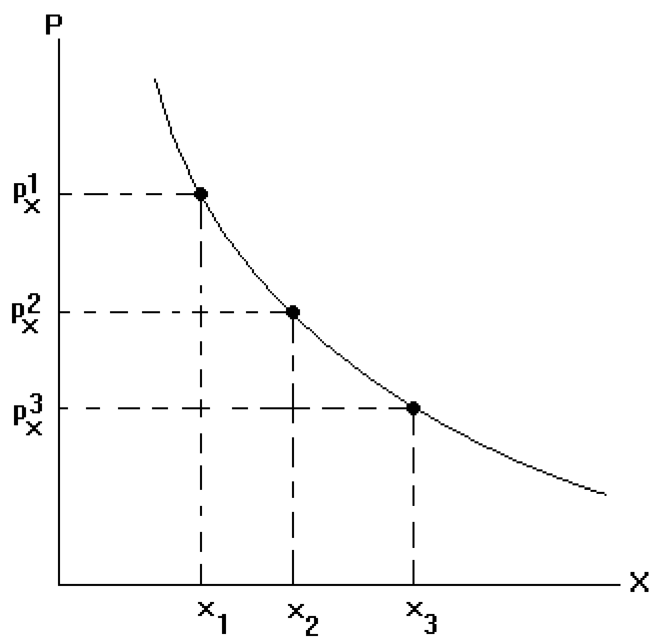
Individual Demand Curves

Demand curves, the relationship between the price of a good and the quantity demanded holding other prices and income constant, can be derived from indifference curve diagrams. Consider how the quantity of X that a person demands changes as the price of Y and income remain constant and the price of X falls from p_x^1 to p_x^2 to p_x^3:



A graph of indifference curves and budget lines showing the quantities of good x demanded for three different prices of x. This graph helps to establish the relationship between indifference curve analysis and demand curves.

The resulting demand curve will involve the prices (p_x^1, p_x^2 and p_x^3) and quantities (x_1, x_2 and x_3) from the indifference curve diagram:



A demand curve graph relating prices and quantities demanded to the previous indifference curve graph.

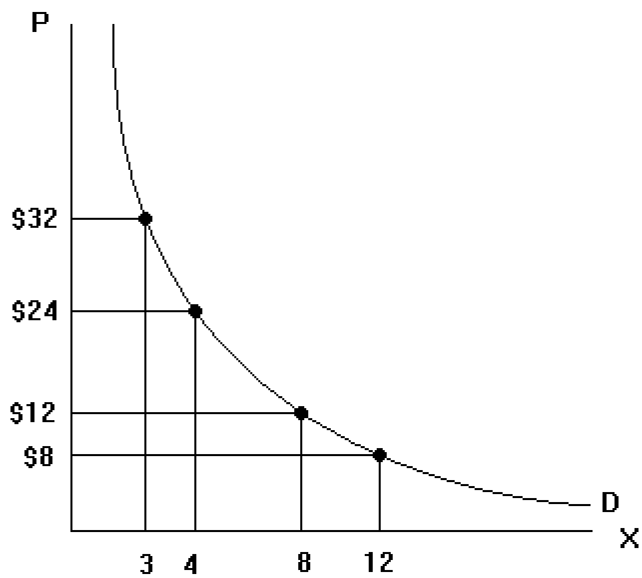
For a math example, imagine that a person has the Cobb-Douglas utility function

$$U = x^{0.4}y^{0.6}$$

It is possible to show, and you should be able to do this, that the demand functions for x and y are:

$$x^* = \frac{0.4I}{p_x} \quad \text{and} \quad y^* = \frac{0.6I}{p_y}.$$

If income, I, is $240, you can diagram the demand curve for x:



A graph showing the demand curve for good x based on the utility function $U = x^{0.4}y^{0.6}$ *and income of $240.*

As the price of X changes, the quantity of X demanded changes according to the demand curve. The demand curve for X doesn't shift when the price of X changes.

So, a change in the price of a good will move a consumer from one point on the demand curve for that good to another point on the same demand curve. This is referred to as a *change in the quantity demanded*. That is, the demand curve itself doesn't change, only the quantity selected on that demand curve changes.

Shifts in the Demand Curve <u>or</u> Changes in Demand (They're the same thing!)

A *change in demand* for a good occurs when something other than the price of that good changes. For example, an increase in income will increase demand for a normal good. This means that the demand curve will shift out (that is up or to the right) and the quantity demanded at any price will increase. An increase in income will decrease the demand for an inferior good. This means that the demand curve will shift in (that is down or to the left) and the quantity demanded at any price will decrease.

A variety of other things might shift the demand curve or change demand. Colder weather will increase the demand for sweaters and mittens, rain will increase demand for umbrellas and salty snacks will increase demand for soft drinks and beer.

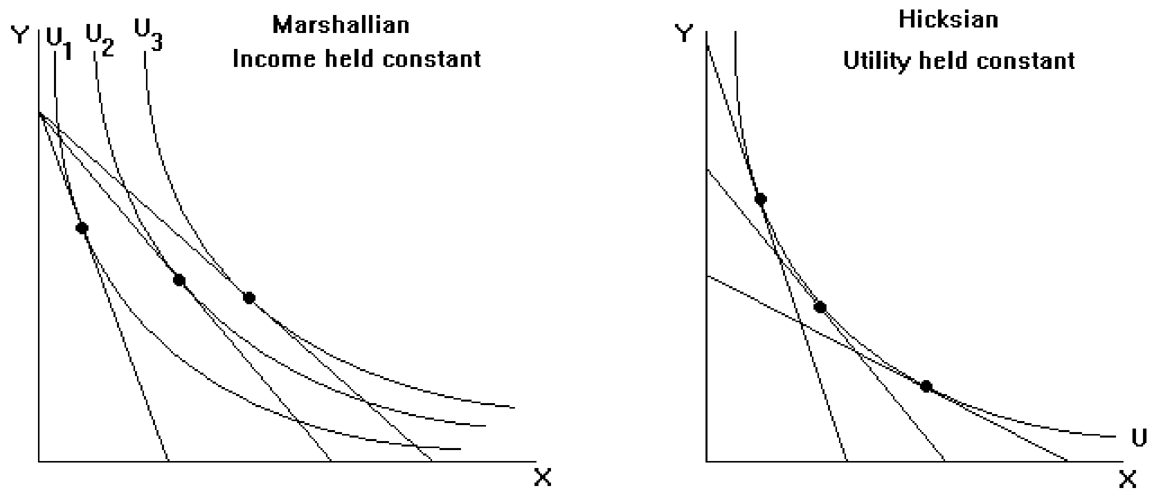
Compensated Demand Curves

The ordinary demand curves discussed above (also known as Marshallian demand curves) are constructed holding income constant and allowing the price of the good to change.

There is an alternative approach to demand curves. The alternative approach constructs demand curves holding utility constant to create *compensated demand curves* (also known as Hicksian demand curves).

Ordinary or Marshallian	income held constant	$x^* = x(p_x, p_y, I)$
Compensated or Hicksian	utility held constant	$x^* = x(p_x, p_y, U)$

The compensated demand curve can be thought of as representing a pure substitution effect with no income effect. That is, instead of rotating about a point on an axis, the budget line related to a compensated demand curve rolls along an indifference curve:

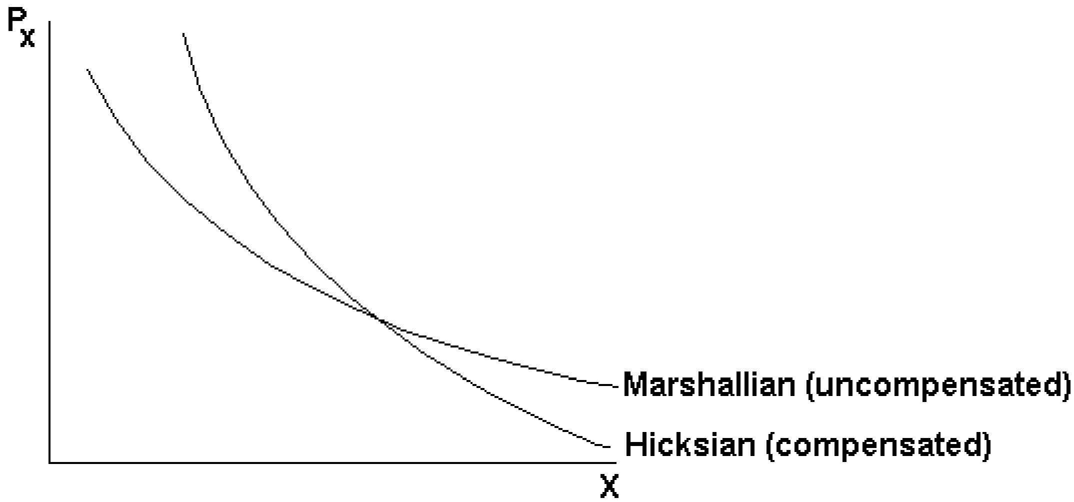


Two graphs of indifference curves and budget lines showing the underlying assumptions of Marshallian and Hicksian demand curves.

The Hicksian demand curve is called the compensated demand curve because the consumer is compensated for the price change. That is, when the price changes they receive compensation that allows them to remain on their original indifference curve. If the price of the good rises this compensation is positive. Of course, if the price falls this compensation is negative.

The effect of this compensation is to reduce the size of changes in the quantity of the good that is consumed. If the price rises, the quantity demanded doesn't fall quite so much because the consumer receives extra money to spend. When the price falls the consumer doesn't increase her consumption quite so much because some money is taken away. The result is that the compensated demand curve is steeper than the ordinary demand curve, reflecting the idea that the changes in the quantity demanded resulting from a price change are smaller.

In a picture, this looks like:



A graph showing the relationship between Marshallian and Hicksian demand curves.

To state this slightly differently, the Marshallian demand curve is more elastic and the Hicksian demand curve is less elastic.

Example 5.3

This example revolves around deriving uncompensated and compensated demand functions and comparing them. It starts with the utility function:

$$U(x,y) = x^{0.5}y^{0.5}$$

The budget constraint is:

$$I = p_x x + p_y y$$

Solving the utility maximization problem gives:

$$\frac{\partial L}{\partial x} = \frac{y^{0.5}}{x^{0.5}} = \lambda p_x$$

$$\frac{\partial L}{\partial y} = \frac{x^{0.5}}{y^{0.5}} = \lambda p_y$$

and taking the ratio of these gives:

$$\frac{y}{x} = \frac{p_x}{p_y} \quad \text{or} \quad p_y y = p_x x$$

and substituting this into the budget constraint gives the demand functions:

$$I = p_x x + p_x x \quad \text{or} \quad x^* = \frac{I}{2p_x} \text{ and } y^* = \frac{I}{2p_y}.$$

Now, if these demand functions are put into the original utility function, we get the indirect utility function that expresses utility as a function of the prices and income:

$$U(x,y) = x^{0.5}y^{0.5} = \left(\frac{I}{2p_x}\right)^{0.5}\left(\frac{I}{2p_y}\right)^{0.5} = \frac{I}{2p_x^{0.5}p_y^{0.5}} = V(I, p_x, p_y)$$

Now, to get the compensated demand functions, we will rewrite the indirect utility function as:

$$V = \frac{I}{2p_x^{0.5}p_y^{0.5}} \quad \text{rewritten as} \quad I = 2Vp_x^{0.5}p_y^{0.5}$$

Plugging this into the demand functions from above gives:

$$x^* = \frac{I}{2p_x} = \frac{2Vp_x^{0.5}p_y^{0.5}}{2p_x} = \frac{Vp_y^{0.5}}{p_x^{0.5}}$$

$$y^* = \frac{I}{2p_y} = \frac{2Vp_y^{0.5}p_x^{0.5}}{2p_y} = \frac{Vp_x^{0.5}}{p_y^{0.5}}$$

These are the compensated demand functions. As discussed in the text, p_y doesn't enter into the uncompensated demand function for x, but it does enter into the compensated demand function for x.

The question is asked in the example, "Are the compensated demand functions homogeneous of degree zero in p_x and p_y if utility is held constant?"

The answer for the functions given here is yes. That is, holding utility (V) constant, if both prices double, then the optimal quantities of x and y do not change.

In addition, this is true for all compensated demand functions. To see this, imagine the utility function:

$$U(x,y) = x^{0.25}y^{0.75}$$

Following the steps above we get:

$$\frac{\partial L}{\partial x} = 0.25\frac{y^{0.75}}{x^{0.75}} = \lambda p_x$$
$$\frac{\partial L}{\partial y} = 0.75\frac{x^{0.25}}{y^{0.25}} = \lambda p_y$$

and the ratio of these is:

$$\frac{y}{3x} = \frac{p_x}{p_y} \quad \text{or} \quad p_y y = 3p_x x$$

and with the budget constraint this is:

$$x^* = \frac{I}{4p_x} \quad \text{and} \quad y^* = \frac{3I}{4p_y}.$$

This gives us:

$$U(x,y) = x^{0.25}y^{0.75} = \left(\frac{I}{4p_x}\right)^{0.25}\left(\frac{3I}{4p_y}\right)^{0.75} = \frac{3^{0.75}I}{4p_x^{0.25}p_y^{0.75}} = V(I, p_x, p_y)$$

$$V = \frac{3^{0.75}I}{4p_x^{0.25}p_y^{0.75}} \quad \text{rewritten as} \quad I = \frac{4Vp_x^{0.25}p_y^{0.75}}{3^{0.75}}$$

Plugging this into the demand functions from above gives:

$$x^* = \frac{I}{4p_x} = \frac{4Vp_x^{0.25}p_y^{0.75}}{3^{0.75}4p_x} = \frac{Vp_y^{0.75}}{3^{0.75}p_x^{0.75}}$$
$$y^* = \frac{3I}{4p_y} = \frac{12Vp_y^{0.75}p_x^{0.25}}{3^{0.75}4p_y} = \frac{3^{0.25}Vp_x^{0.25}}{p_y^{0.25}}$$

Both of these demand functions are homogeneous of degree zero in the prices, holding utility constant. That is, if both prices double (for example) and utility is held constant, the quantities of x and y that are demanded will not change.

A Mathematical Development of the Response to Price Changes and the Slutsky Equation

The response to a change in the price of x will be a change in the quantity of x demanded. This movement along the demand curve (and not a shift in the demand curve) is represented by the partial derivative:

$$\frac{\partial x}{\partial p_x}\ .$$

The direct approach to calculating this in the most general situation is to set up the Lagrangian:

$$L = U(x_1, x_2, \ldots x_n) + \lambda(I - p_1x_1 - p_2x_2 - \ldots - p_nx_n)$$

and take the n+1 partial derivatives (the partial derivative with respect to each good, x_i plus the derivative with respect to λ) to get n+1 equations, which can then be solved simultaneously. However, this is a lot of work and, in general, these n+1 equations may not be solvable.[1]

Let us try an indirect approach and see what we get.

We'll start with two goods, cleverly names x and y, and the compensated demand function:

$$x^c(p_x, p_y, U)$$

which expresses the quantity of x that will be demanded for given prices of x and y and a given utility level.

The relationship between the ordinary demand function,

$$x(p_x, p_y, I)$$

and the compensated demand function may be derived by thinking of a minimum expenditure function that described the minimum expenditure necessary to achieve a given utility level:

$$E(p_x, p_y, U)$$

then we can define the compensated demand function to be:

$$x^c(p_x, p_y, U) = x(p_x, p_y, E(p_x, p_y, U))$$

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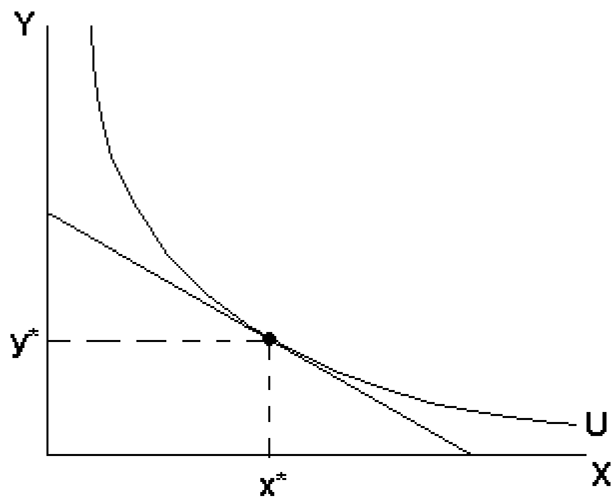
[1] A nasty fact that they don't usually tell you in math classes is that most mathematical problems don't actually have an analytical solution.

where the income level is replaced by the minimum necessary expenditure in the regular, uncompensated, demand function.

Now, to come up for air for just a minute, these are all different ways of expressing the basic idea of the standard utility maximization graph:



A graph showing an optimal consumption bundle from a consumer's utility maximization problem.

The optimal bundle, (x^*,y^*) can be thought of as maximizing utility given the budget constraint, yielding $x(p_x, p_y, I)$ and the indirect utility function $V(p_x, p_y, I)$. It may also be thought of as minimizing the expenditure needed to achieve the given utility level, yielding $x^c(p_x, p_y, U)$ and the expenditure function $E(p_x, p_y, U)$.

To jump ahead a bit, these can be set up as two ways of looking at the same problem, either utility maximization given income or as expenditure minimization given utility.

Primal	Dual
maximize U(x,y) subject to $I = p_x x + p_y y$	minimize E(x,y) subject to U=U(x,y)
Indirect utility function $V(p_x, p_y, I)$	Expenditure function $E(p_x, p_y, U)$
Marshallian demand functions $x(p_x, p_y, I)$	Compensated demand functions $x^c(p_x, p_y, U)$

It should also be noted that the indirect utility function and expenditure function are inverses of each other. That is, if you solve the indirect utility function for I you get the expenditure function. If you solve the expenditure function for U, you get the indirect utility function.

OK, back to the show. We had the following relationship between the compensated and uncompensated demand functions:

$$x^c(p_x, p_y, U) = x(p_x, p_y, E(p_x, p_y, U))$$

Now, if we take the partial derivative with respect to p_x on each side of the equation, we get:

$$\frac{\partial x^c}{\partial p_x} = \frac{\partial x}{\partial p_x} + \frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x}$$

which can be rewritten as:

$$\frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - \frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x}$$

This represents the change in ordinary, uncompensated or Marshallian quantity demanded $\left(\frac{\partial x}{\partial p_x}\right)$ as the combination of two effects, a substitution effect $\left(\frac{\partial x^c}{\partial p_x}\right)$ and an income effect $\left(\frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x}\right)$.

The substitution effect is the change in quantity demanded of x along the original indifference curve. This is exactly $\frac{\partial x^c}{\partial p_x}$.

Let's look at the income effect a bit more. First, because expenditures (E) are equivalent to income (I) at an optimal solution (and they're two sides of the same thing) we can replace $\frac{\partial x}{\partial E}$ with $\frac{\partial x}{\partial I}$.

Second, the curious term $\frac{\partial E}{\partial p_x}$ is actually very familiar. This is the partial derivative of expenditures (holding utility constant) with respect to the price of x. Now, imagine that you are consuming five units of x and the price of x rises by $1. By how much will expenditures on x rise? At the margin, the answer is $5. This partial derivative is actually the quantity of x that you are consuming, known as x. So, we can replace $\frac{\partial E}{\partial p_x}$ with x to get the income effect:

$$x\frac{\partial x}{\partial I}$$

Now, the entire function from above can be rewritten as the Slutsky equation[2], which relates the change in the Marshallian quantity demanded as a combination of the substitution and income effects:

$$\frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - x\frac{\partial x}{\partial I} \quad \text{or} \quad \frac{\partial x}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U=\overline{U}} - x\frac{\partial x}{\partial I}$$

Now, to consider whether each effect is positive or negative:

The substitution effect is always negative.

The sign of the income effect depends on the sign of $\frac{\partial x}{\partial I}$, or whether the good is a normal good $\left(\frac{\partial x}{\partial I} > 0\right)$ or is an inferior good $\left(\frac{\partial x}{\partial I} < 0\right)$. Usually, the net effect will be that $\frac{\partial x}{\partial p_x} < 0$. However, if the good is an inferior good and if the magnitude of the income effect is greater than the magnitude of the substitution effect, we get the odd result that quantity demanded rises as the price rises, the case of the so-called Giffen good.

So, to put this together a bit:

Type of good	**Slutsky Interpretation**
Normal good	$\frac{\partial x^c}{\partial p_x} < 0$ $x\frac{\partial x}{\partial I} > 0$

[2] I am morally compelled to point out that Eugenio Slutsky was a great statistician as well as a great economist. He also has a very important statistics relationship with his name attached to it. My friends and I like to pretend that he was also a multi-sport athlete in college, but pursued academia rather than professional sports due to an injury near the end of his senior year.

	$\frac{\partial x}{\partial p_x} < 0$
Inferior good	$\frac{\partial x^c}{\partial p_x} < 0$ $x\frac{\partial x}{\partial I} < 0$, but smaller than $\frac{\partial x^c}{\partial p_x}$ $\frac{\partial x}{\partial p_x} < 0$
Giffen good	$\frac{\partial x^c}{\partial p_x} < 0$ $x\frac{\partial x}{\partial I} < 0$ and larger than $\frac{\partial x^c}{\partial p_x}$ $\frac{\partial x}{\partial p_x} > 0$

Example 5.4

If we start with the utility function:

$$U(x,y) = x^{0.5}y^{0.5}$$

we get the Marshallian (uncompensated) and Hicksian (compensated) demand functions:

$$x(p_x, p_y, I) = \frac{I}{2p_x} \quad \text{and} \quad x^c(p_x, p_y, V) = \frac{Vp_y^{0.5}}{p_x^{0.5}}$$

The partial derivative of the Marshallian demand function with respect to p_x is:

$$\frac{\partial x}{\partial p_x} = \frac{-I}{2p_x^2}$$

Now, we can think about the substitution effect and the income effect.

The substitution effect is the partial derivative of the compensated demand function with respect to the price of x:

$$\frac{\partial x^c}{\partial p_x} = \frac{-Vp_y^{0.5}}{2p_x^{1.5}}$$

but $V = \frac{I}{2p_x^{0.5}p_y^{0.5}}$, giving us:

$$\frac{\partial x^c}{\partial p_x} = \frac{-\frac{I}{2p_x^{0.5}p_y^{0.5}}p_y^{0.5}}{2p_x^{1.5}} = \frac{-I}{4p_x^2} \quad \text{(substitution effect)}$$

The income effect is equal to the product of x and $\frac{\partial x}{\partial I}$, or:

$$x\frac{\partial x}{\partial I} = x\frac{1}{2p_x} = \frac{x}{2p_x}$$

But x is given by the demand function $x = \frac{I}{2p_x}$, so we have:

$$x\frac{\partial x}{\partial I} = \frac{x}{2p_x} = \frac{\frac{I}{2p_x}}{2p_x} = \frac{I}{4p_x^2}$$

And the entire Slutsky equation can be written as:

$$\frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - x\frac{\partial x}{\partial I}$$

$$\frac{\partial x}{\partial p_x} = \frac{-I}{4p_x^2} - \frac{I}{4p_x^2} = \frac{-2I}{4p_x^2}$$

As pointed out in the book, for the Cobb-Douglas utility function (a utility function of the form $U = x^a y^{1-a}$) the substitution and income effects are of the same magnitude. This is not true for all utility functions.

Demand Elasticities

Elasticities are popular in economics. We'll talk about three different elasticities here. Note that as the equations are written here they are based on Marshallian (normal or uncompensated) demand curves.

1. Price elasticity of demand describes the relationship between the percentage change in the price of a good and the resulting percentage change in the quantity demanded. This can be expressed in a variety of ways:

$$e_{x,p_x} = \frac{\%\Delta x}{\%\Delta p_x} = \frac{\frac{\Delta x}{x}}{\frac{\Delta p_x}{p_x}} = \frac{\Delta x}{\Delta p_x} \cdot \frac{p_x}{x} = \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{x}$$

It is also equal to the inverse of the slope of the demand curve multiplied by $\frac{p_x}{x}$.

2. Income elasticity of demand expresses the relationship between percentage change in income and the percentage change in the demand for the good. This can also be expressed in several ways:

$$e_{x,I} = \frac{\%\Delta x}{\%\Delta I} = \frac{\frac{\Delta x}{x}}{\frac{\Delta I}{I}} = \frac{\Delta x}{\Delta I} \cdot \frac{I}{x} = \frac{\partial x}{\partial I} \cdot \frac{I}{x}$$

If this is positive, the good is a normal good. If this is negative, the good is an inferior good.

3. Cross price of elasticity expresses the relationship between the percentage change in the price of one good and the percentage change in demand for another good. This can also be expressed in several ways:

$$e_{x,p_y} = \frac{\%\Delta x}{\%\Delta p_y} = \frac{\frac{\Delta x}{x}}{\frac{\Delta p_y}{p_y}} = \frac{\Delta x}{\Delta p_y} \cdot \frac{p_y}{x} = \frac{\partial x}{\partial p_y} \cdot \frac{p_y}{x}$$

Price elasticity of demand

Price elasticity of demand (sometimes called *own price elasticity of demand* to distinguish it from cross price elasticity) is percent change in quantity demand divided by percent change in price. This is usually a negative number.

If the elasticity is less than –1 (-1.5 or –2.0, for example) then demand for the good is said to be *elastic*. Put somewhat simply, this is consistent with consumers being very flexible or elastic in their demand for the good, so that if the price rises even slightly, they can easily substitute away from it. If elasticity is –4, for example, when the price rises 5% the quantity demanded will fall by 20%.

If the elasticity is greater than –1 (-0.8 or –0.5, for example) then demand for the good is said to be *inelastic*. This is consistent with consumers being very inflexible or inelastic in their demand for the good, so that even if the price rises a lot they cannot easily reduce their consumption. If elasticity is –0.1, for example, when the price rises by 20% the quantity demand will fall by only 2%.

If demand is elastic, when the price rises total expenditures on the good will fall and when price falls total expenditures on the good will rise.

If demand is inelastic, when the price rises total expenditures will rise and when price falls total expenditures will fall.

This can be established using calculus.

Total expenditures are $x \cdot p_x$. Taking the derivative of this with respect to p_x gives us:

$$\frac{\partial(x \cdot p_x)}{\partial p_x} = p_x \frac{\partial x}{\partial p_x} + x \frac{\partial p_x}{\partial p_x}$$

$$= \frac{x}{x} p_x \frac{\partial x}{\partial p_x} + x \cdot 1$$

$$= x\left(\frac{p_x}{x} \frac{\partial x}{\partial p_x} + 1 \right)$$

$$= x\left(e_{x,p_x} + 1\right)$$

So the change in expenditures resulting from a change in price depends on the quantity consumed (x), and the sum of the elasticity and one.

If demand for the good is elastic, $e_{x,p_x} < -1$ and the term in parentheses is negative, so when price rises expenditures fall.

If demand for the good is elastic, $e_{x,p_x} > -1$ and the term in parentheses is positive, so when price rises expenditures rise.

If demand is *unit elastic*, then $e_{x,p_x} = -1$ and total expenditures remain the same regardless of the price.

Compensated Demand Elasticities

Whereas before we were talking about Marshallian (uncompensated or ordinary) demand elasticities, we will now consider Hicksian (compensated) demand elasticities.

To some extent the difference is only notational, as the ordinary demand functions are replaced by the compensated demand functions:

$$e_{x^c,p_x} = \frac{\%\Delta x^c}{\%\Delta p_x} = \frac{\frac{\Delta x^c}{x^c}}{\frac{\Delta p_x}{p_x}} = \frac{\Delta x^c}{\Delta p_x}\cdot\frac{p_x}{x^c} = \frac{\partial x^c}{\partial p_x}\cdot\frac{p_x}{x^c}$$

$$e_{x^c,p_y} = \frac{\%\Delta x^c}{\%\Delta p_y} = \frac{\frac{\Delta x^c}{x^c}}{\frac{\Delta p_y}{p_y}} = \frac{\Delta x^c}{\Delta p_y}\cdot\frac{p_y}{x^c} = \frac{\partial x^c}{\partial p_y}\cdot\frac{p_y}{x^c}$$

How different these are from the uncompensated elasticities depends on how big the income effect is. The uncompensated elasticities include income effects while the compensated elasticities are based only on substitution effects.

The difference can be seen by applying the Slutsky equation to these elasticities.

$$e_{x,p_x} = \frac{\partial x}{\partial p_x}\cdot\frac{p_x}{x} = \frac{p_x}{x}\cdot\left[\frac{\partial x^c}{\partial p_x} - x\frac{\partial x}{\partial I}\right] = \frac{p_x}{x}\cdot\frac{\partial x^c}{\partial p_x} - p_x\frac{\partial x}{\partial I}$$

So, the difference between the compensated and uncompensated price elasticity of demand is $p_x \frac{\partial x}{\partial I}$. Put somewhat differently,

$$\text{Uncompensated Elasticity} = \text{Compensated Elasticity} - p_x\frac{\partial x}{\partial I}$$

$$UE = CE - p_x\frac{\partial x}{\partial I}\cdot\frac{I}{I}\cdot\frac{x}{x}$$

$$UE = CE - p_x\cdot\frac{x}{I}\cdot\frac{\partial x}{\partial I}\cdot\frac{I}{x}$$

$$UE = CE - \frac{p_x\cdot x}{I}\cdot e_{x,I}$$

So, the uncompensated elasticity is equal to the compensated elasticity minus the product of $\frac{p_x \cdot x}{I}$ and $e_{x,I}$.

$e_{x,I}$ is the income elasticity of demand.

$\frac{p_x \cdot x}{I}$ is the total expenditure on x divided by income. This is also known as the share of income spent on good x, which goes by the name s_x. Thus we get:

$$UE = CE - s_x \cdot e_{x,I}$$

So the difference between the uncompensated and compensated elasticities for a good is the product of the share of income spent on the good and the income elasticity of demand for the good.

The difference between the two will be small if a small share of income is spent on the good (like with salt or chewing gum) or if the income elasticity of demand is small. If either or both of these are true, then the difference between compensated and uncompensated elasticities will be negligible.

Relationships Between Goods and Elasticities

I will present these without proof or a lot of discussion. Just sort of hold on to them for now and don't lose any sleep over them.

Homogeneity of elasticities for a good

The sum of the own price elasticity, cross price elasticities and income elasticity of demand for a good is zero:

$$e_{x,p_x} + e_{x,p_y} + e_{x,I} = 0$$

Engel aggregation

The sum of the products of income shares of each good and the income elasticity of each good is one:

$$s_x e_{x,I} + s_y e_{y,I} = 1$$

Cournot aggregation

Cournot aggregation is a statement about how the own price elasticity and the cross price elasticity are constrained by the budget constraint:

$$s_x e_{x,p_x} + s_y e_{y,p_x} = -s_x$$

An Endorsement for Example 5.5

You should go through example 5.5 in the text and be sure that you can work from the original utility functions to get the demand functions and elasticities discussed in this example. To be clear about this, you should be able to get demand functions for x and y from the following:

$U(x,y) = x^{\alpha}y^{1-\alpha}$ $\qquad p_x x + p_y y = I$

$U(x,y) = x^{0.5} + y^{0.5}$ $\qquad p_x x + p_y y = I$

$U(x,y) = -x^{-1} - y^{-1}$ $\qquad p_x x + p_y y = I$

The first is a Cobb-Douglas utility function and the remaining two are constant elasticity of substitution (CES) utility functions. As a safety tip, it might not be possible for the third utility function.

Consumer Surplus

Consumer surplus is the gain from trade accruing to the consumer or consumers in a market. In a principles of microeconomics sense, this is equal to the difference between a consumer's total willingness to pay for a good and their total expenditure on that good. Put somewhat differently, the consumer surplus from a good at a price is the area under the Marshallian demand curve above the price. It is the maximum amount that an individual consumer would pay for the right to make voluntary purchases at that price.

From the point of view of indifference curves and budget lines, there are a couple of different approaches to consumer surplus.

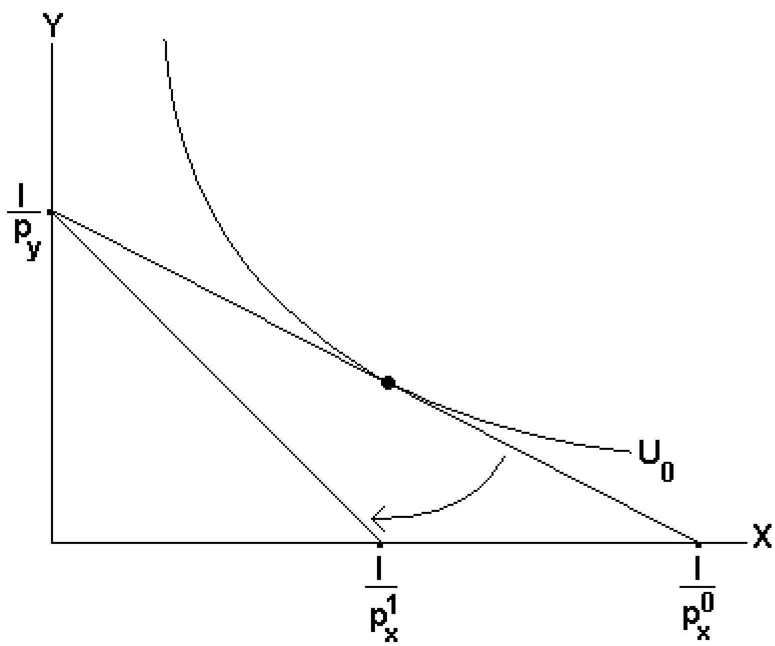
In terms of expenditures, we can think about the cost to a consumer of an increase in the price of one good, holding the price of the other good and utility level constant. That is, when the price of one good rises, how much more do you have to spend to maintain your initial level of utility? This measure of the change in consumer welfare or the change in consumer surplus is called the *compensating variation (CV)*.

Mathematically, this can be expressed in terms of the expenditure function as:

$$CV = E\left(p_x^1, p_y, U_0\right) - E\left(p_x^0, p_y, U_0\right)$$

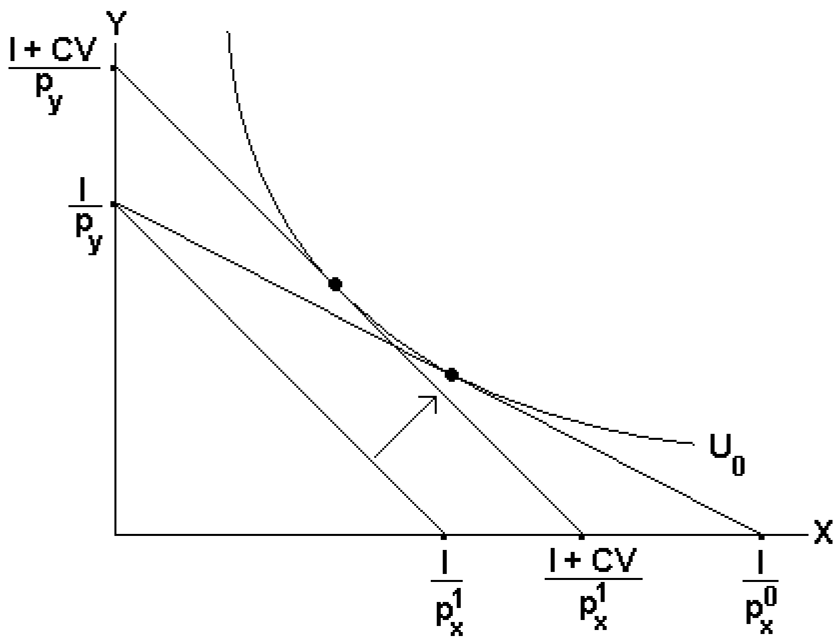
If the price of good x rises from p_x^0 to p_x^1, then expenditures will rise. This increase in expenditures is the compensating variation. a measure of the loss in consumer welfare caused by the price increase. Put slightly differently, this is the additional income that the consumer would need in order to preserve her initial level of utility, U_0.

In terms of a graph, this looks like the following. First, we start out with income I and prices of p_x^0 and p_y. Then the price of good x rises to p_x^1, represented by the rotation of the budget line as shown below.



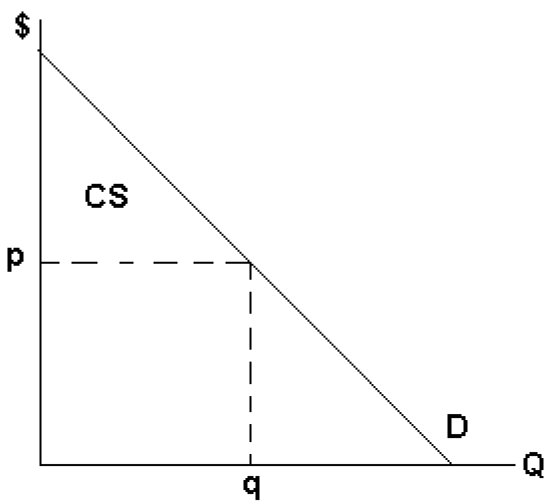
A graph showing the impact on the budget line of an increase in the price of good x.

The problem is that the increase in the price of x puts the consumer on a lower indifference curve. The compensating variation (CV) is the additional income that would be needed to bring the consumer back up to the original utility level (U_0), given the new, higher price of x. This is shown below:



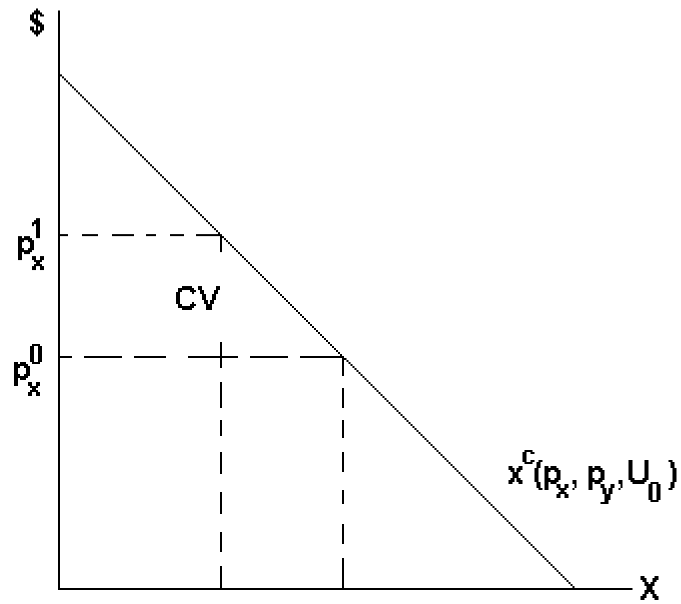
A graph showing the compensating variation associated with an increase in the price of good x. This is the additional income that would be necessary to make a person as well off after the price increase as she was before the price increase.

Now, it might help to have this tied back to the usual picture of consumer surplus from the demand curve, which sort of looks like this:



A graph showing consumer surplus on a demand graph.

Now we can do better. Consumer surplus is most precisely defined using the compensated demand curve, rather than the Marshallian demand curve. We can use the compensating variation from above as a measure of the change in consumer surplus when the price of the good x rises from p_x^0 to p_x^1, holding utility constant. This is:

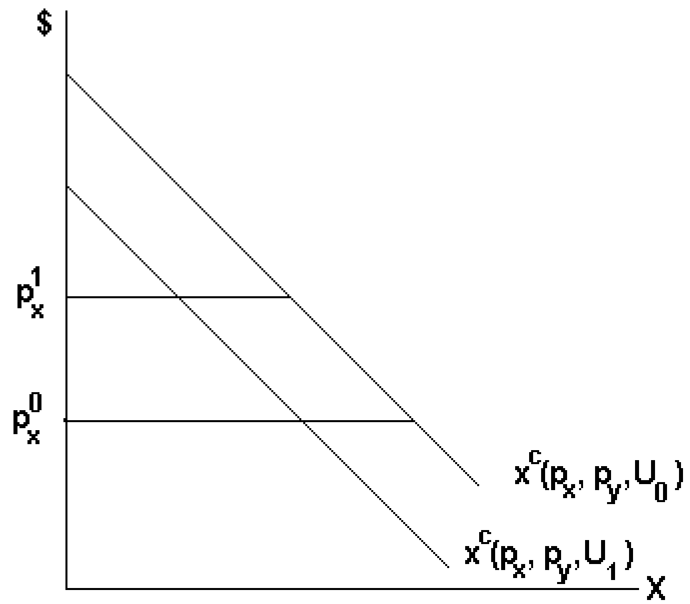


A graph showing compensating variation associated with a price change and a Hicksian (compensated) demand curve.

Marshallian versus Compensated Consumer Surplus

There's a nasty mess of ideas presented in the book in Figures 5.8 and 5.9. They relate to two different measures of consumer surplus. These are the *compensating variation*, which is discussed above, and the *equivalent variation*, which I will attempt to explain here.

Looking at Figure 5.8(a), there are two utility levels, U_0 and U_1. U_0 represents the higher utility level. These can be related to two compensated demand curves for good x:

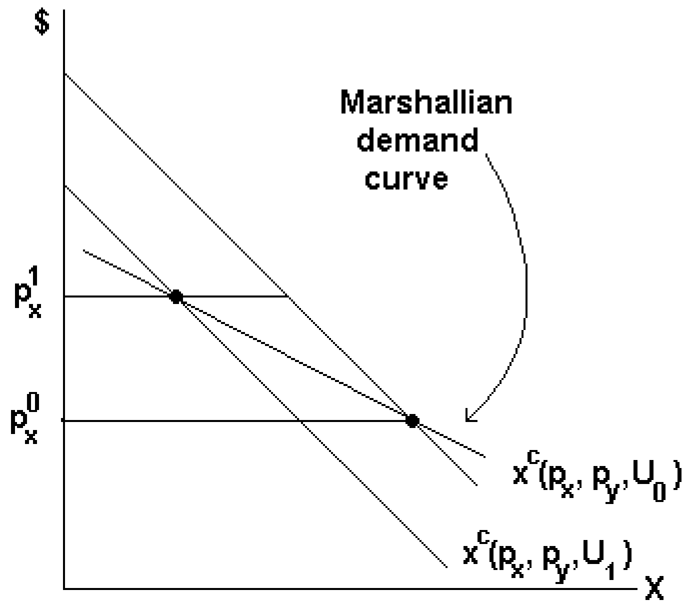


A graph showing that there are two different compensated demand curves when the price of a good changes, the compensated demand curve associated with the initial price of the good and the compensated demand curve associated with the new price of the good.

The question is, which of the two compensated demand curves should be used in calculating the change in consumer surplus resulting from the price increase from p_x^0 to p_x^1? The answer really depends on what you believe the consumer has a right to. If you believe that the consumer has a right to the initial, lower price of x, then the loss of consumer surplus is the larger area and uses the U_0 demand curve to calculate lost consumer surplus. This is the amount of compensation, the compensating variation, that the consumer would need to bring her back up to her original utility level. If you believe that the consumer doesn't have a right to the initial price, then the loss of consumer surplus is the smaller area and uses the U_1 demand curve to calculate lost consumer surplus. This is the amount that she would be willing to pay, the equivalent variation, to face the lower price for x instead of the higher price, given that she starts out facing the higher price.

So, the difference between compensating variation (CV) and equivalent variation (EV) is that CV represents a consumers willingness to accept, or the compensation they would require to be compensated for a price increase, whereas the EV represents willingness to pay to prevent a price increase. The difference between the two really depends on the income effect of the price change.

Now, the Marshallian or uncompensated demand curve is the quantity actually demanded at each of the two prices and is sort of a compromise between the two compensated demand curves. The consumer surplus calculated on the Marshallian demand curve will be in between the calculations from the two compensated demand curves.



A graph showing the relationship between the Marshallian and Hicksian (compensated) demand curves.

An Example

To see the difference between CV and EV, imagine a utility function U(x,y) = xy.

To calculate the CV, imagine that we start with p_x=1 and p_y=1 and I=100. At a utility maximizing bundle, we have x=50, y=50 and U=2500. If the price of x rises so that p_x=2, achieving that same level of utility, U=2500, will now require a higher level of income. This can be calculated from the expenditure function E(p_x, p_y, U), which turns out to be given by:

$$E(p_x, p_y, U) = 2p_x^{0.5}p_y^{0.5}U^{0.5}$$

So, in the original case with p_x=1 and p_y=1 and U=2500, we had I=100.

When the price of x rises to 2, to maintain the original utility level we have p_x=2 and p_y=1 and U=2500 and:

$$E(2,1,2500) = 2 \cdot 2^{0.5} \cdot 1^{0.5} \cdot 2500^{0.5} = 141.421$$

So, the compensating variation is the extra income needed to achieve the original utility level at the new prices, or $41.421.

To calculate the EV, we will consider what this person's utility level would be if the price of x were to rise to p_x=2, assuming their income stayed at I=100. This is based on the indirect utility function V(p_x, p_y, I), which turns out to be given by:

$$V(p_x, p_y, I) = \frac{I^2}{4p_x p_y}$$

When the price of x rises to 2 and income is at I=100 we have:

$$V(2,1,100) = \frac{100^2}{4 \cdot 2 \cdot 1} = 1250$$

At the original prices, this utility level would be achieve with an income of:

$$E(1,1,1250) = 2 \cdot 1^{0.5} \cdot 1^{0.5} \cdot 1250^{0.5} = 70.711$$

The difference between this and the original income level, 100 – 70.711 = 29.289. So, the equivalent variation is income that this person would be willing to give up to avoid the price increase, or $29.289.

Some Exercises

For a standard diagram with indifference curves and budget lines, show the following.
1. An increase in the price of x.
2. An increase in the price of y.
3. An income increase with both goods normal.
4. An income increase with x inferior.
5. An income increase with y inferior.

For the following price changes, show the substitution and income effects.
6. Increase in the price of x with both goods normal.
7. Increase in the price of x with x inferior.
8. Increase in the price of x with y inferior.
9. Decrease in the price of x with both goods normal.
10. Decrease in the price of x with x inferior.
11. Decrease in the price of x with y inferior.
12. Increase in the price of y with both goods normal.
13. Increase in the price of y with x inferior.
14. Increase in the price of y with y inferior.

15. Decrease in the price of y with both goods normal.
16. Decrease in the price of y with x inferior.
17. Decrease in the price of y with y inferior.

18. What does it mean to say that Marshallian demand functions are homogeneous of degree zero in prices and income?

For each of the following, indicate whether the good described is a normal good, an inferior good or a Giffen good.
19. Substitution effect dominates the income effect.
20. Substitution effect dominates the income effect, but the income effect of a price increase is positive.
21. Income effect dominates the substitution effect, but the income effect of a price increase is positive.

Calculate the uncompensated (Marshallian) and compensated (Hicksian) demand functions for x and y for the following Cobb-Douglas utility functions.
22. $U(x,y) = x^{0.5}y^{0.5}$
23. $U(x,y) = x^{0.1}y^{0.9}$
24. $U(x,y) = x^{0.2}y^{0.8}$
25. $U(x,y) = x^{0.3}y^{0.7}$
26. $U(x,y) = x^{0.4}y^{0.6}$
27. $U(x,y) = x^{0.6}y^{0.4}$
28. $U(x,y) = x^{0.7}y^{0.3}$
29. $U(x,y) = x^{0.8}y^{0.2}$
30. $U(x,y) = x^{0.9}y^{0.1}$

For each of the following utility functions given in questions 22-30, write out the entire Slutsky equation $\frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - x\frac{\partial x}{\partial I}$.